Mode I and Mode II Analyses of a Crack Normal to the Graded Interlayer in Bonded Materials

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In this paper, the plane elasticity equations are used to investigate the in-plane normal (mode I) and shear (mode II) behavior of a crack perpendicular to and terminating at the interface in bonded media with a graded interfacial zone. The interfacial zone is treated as a nonhomogeneous interlayer with the continuously varying elastic modulus between the two dissimilar, homogeneous semi-infinite constituents. For each of the individual loading modes, based on the Fourier integral transform technique, a singular integral equation with a Cauchy kernel is derived in a separate but parallel manner. In the numerical results, the values of corresponding modes of stress intensity factors are illustrated for various combinations of material and geometric parameters of the bonded media in conjunction with the effect of the material nonhomogeneity within the graded interfacial zone.

Key Words: Bonded Materials, Functionally Graded Materials, Nonhomogeneous Interlayer, Singular Integral Equations, Stress Intensity Factors

1. Introduction

One of the important problems to be carefully noted with using the conventional bonded materials is the inevitable existence of the sharp interface between the dissimilar constituents where the apparent property mismatch prevails. As a result, the interface has usually been viewed as a vulnerable area subjected to generally high thermal and residual stresses and relatively weak bonding strength with the higher likelihood of failure. From the standpoint of analytical fracture mechanics, the ideal interface modeling employed in solving crack problems for such piecewise homogeneous media also suffers from the anomalous crack-tip behavior of complex power or nonsquare-root singularities, depending on the crack locations. Related studies belonging to this category of crack problems, for instance, can be attributed to Rice (1988) and Romeo and Ballarini (1995) for the cases of a crack lying along and terminating perpendicularly at the interface, respectively, and further typical references quoted therein.

The recent development of the concept of functionally graded materials featuring the spatial variations of the properties, however, paved the way to alleviate the aforementioned physical shortcomings arising from the stepwise change in the material parameters of bonded or layered media (Lee and Erdogan, 1994; Suresh and Mortensen, 1997). Such a new class of materials, for example, would viably be utilized as interfacial zones to bond basically incompatible dissimilar materials, as thermal barrier coatings to replace homogeneous coatings, and as wear resistant layers in various components such as bearings, gears, cams and other machine tools (Koizumi, 1993). Moreover, with the fracture mechanics-based applications of the graded materials in mind, a series of analytical solutions to some crack problems has been obtained on a nonhomogeneous continuum basis which is well reviewed by Erdogan (1995). Specifically,

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assuming the material nonhomogeneity expressed as the exponential variation of the elastic modulus, Delale and Erdogan (1988) proposed the square-root character of the crack-tip field when the crack lies along the interface with the nonhomogeneous half-plane. For a crack perpendicular to and intersecting the interface with the nonhomogeneous material, Erdogan et al. (1991a, b) showed that the square-root singular behavior also prevails under mode I and mode III loading conditions, respectively. Schovanec and Walton (1988), Martin (1992), and Jin and Noda (1994) verified that the above conjecture is true for the general material nonhomogeneity, provided the variation of elastic properties in the spatial domain is continuous and piecewise differentiable near and at the crack tip. In particular, Jin and Noda (1994) showed that the corresponding near-tip field quantities possess the same angular distributions around the crack tip as those for homogeneous materials. The influence of material gradients in the vicinity of the crack tip thus manifests itself only through the crack driving forces such as the stress intensity factors. Additional recent contributions pertinent to the crack problems entailing graded properties are, among others, due to Jin and Batra (1996), Chen and Erdogan (1996), Bao and Cai (1997), and Shbeeb and Binienda (1999) who considered the delamination or interface cracking between the graded coating and the substrate, whereas the problems of a mixed mode crack and an array of parallel cracks in a functionally graded plane were resolved by Konda and Erdogan (1994) and Choi (1997), respectively. Besides, the finite element studies of fracture of functionally graded materials were conducted by Gu et al. (1999) and Anlas et al. (2000).

The objective of this paper is to provide the plane elasticity solutions for both the mode I and mode II problems of a crack perpendicular to the graded interfacial zone in bonded dissimilar, homogeneous half-planes. The interfacial zone is treated as a nonhomogeneous interlayer with the continuous variation of its elastic modulus in the thickness direction. To be pointed out is that as discussed in the foregoing, Erdogan et al. (1991a) investigated a similar problem of a mode I crack where the graded material was assumed to be semi-infinite. In consideration of the primary use of the graded materials, however, a more interesting and realistic model that would find more applications in practice is to incorporate the presence of such graded materials in the form of an interlayer between the adjoining dissimilar constituents, as was examined for a relatively simple case of a mode III crack by Erdogan et al. (1991b). The formulation of the crack problem, based on the Fourier integral transform technique, is reduced to solving a Cauchy-type singular integral equation for each of the individual loading modes. For applying the fracture mechanics criteria, the main results presented are the mode I and mode II stress intensity factors evaluated from the near-tip stress fields with the square-root singularity, which are then illustrated as functions of material and geometric parameters of the bonded media with the graded interfacial zone.

2. Problem Statement and Formulation

Consider the two dissimilar, homogeneous media bonded through a graded interfacial zone. As shown in Fig. 1, the medium on the right-hand side contains a crack of length $2c=b-a$ and location d perpendicular to the interfacial zone and let $(x, y)=(x_k, y)$, $k=1, 2, 3$, be the local coordinates with their origins at the locations of nominal interfaces. It is assumed that the dimensions of the homogeneous constituents

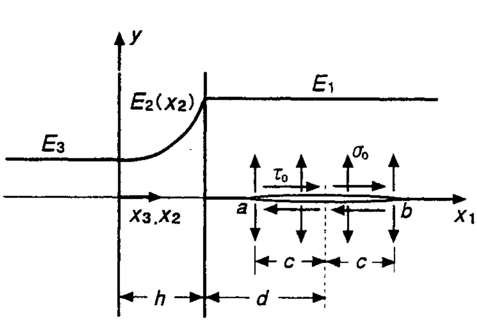

Fig. 1 Configuration, coordinate systems, and loading conditions for cracked dissimilar media bonded through a graded interfacial zone

are considerably greater than that of the interfacial zone and the crack length. Hence, such constituents are modeled to be semi-infinite. By denoting the elastic moduli of the homogeneous media as E_k, $k=1, 3$, and treating the interfacial zone as a nonhomogeneous interlayer of finite thickness h, the elastic modulus of the interlayer $E_2(x)$ is assumed to follow an exponential variation as (Erdogan, 1995)

$$E_2(x)=E_3 e^{\beta x},\ \beta=\frac{1}{h}\ln\left(\frac{E_1}{E_3}\right) \tag{1}$$

where in the local coordinate $(x, y)=(x_2, y)$, the nonhomogeneity parameter β as specified above allows the continuous transition of the elastic moduli from one half-plane to the other. Whereas the Poisson's ratio is simply taken to be $\nu_k=\nu=$ constant, $k=1,2,3$, which is justifiable to the extent that the influence of ν on the stress intensity factors is rather insignificant (Konda and Erdogan 1994; Choi, 1997).

With $u_k(x,y)$ and $v_k(x,y)$, $k=1,2,3$, referring to the x- and y-components of displacements, respectively, the constitutive relations are given as

$$\sigma_{kxx}=\frac{\mu_k}{\kappa-1}\left[(1+\kappa)\frac{\partial u_k}{\partial x}+(3-\kappa)\frac{\partial v_k}{\partial y}\right] \tag{2}$$

$$\sigma_{kyy}=\frac{\mu_k}{\kappa-1}\left[(1+\kappa)\frac{\partial v_k}{\partial y}+(3-\kappa)\frac{\partial u_k}{\partial x}\right] \tag{3}$$

$$\sigma_{kxy}=\mu_k\left(\frac{\partial u_k}{\partial y}+\frac{\partial v_k}{\partial x}\right);\ k=1,\ 2,\ 3 \tag{4}$$

where $\sigma_{klm}(x, y)$, $k=1, 2, 3$, $l,m=x, y$, are the stress components, $\mu_k=E_k/2(1+\nu)$, $k=1, 2, 3$, are the shear moduli, and $\kappa=3-4\nu$ for plane strain state and $\kappa=(3-\nu)/(1+\nu)$ for plane stress state. In the absence of body forces, a system of governing equations corresponding to Eq. (1) can be written as

$$\nabla^2 u_k+\frac{2}{\kappa-1}\left(\frac{\partial^2 u_k}{\partial x^2}+\frac{\partial^2 v_k}{\partial x\partial y}\right)+\frac{\beta}{\kappa-1}\left[(1+\kappa)\frac{\partial u_k}{\partial x}+(3-\kappa)\frac{\partial v_k}{\partial y}\right]=0 \tag{5}$$

$$\nabla^2 v_k+\frac{2}{\kappa-1}\left(\frac{\partial^2 u_k}{\partial x\partial y}+\frac{\partial^2 v_k}{\partial y^2}\right)+\beta\left(\frac{\partial u_k}{\partial y}+\frac{\partial v_k}{\partial x}\right)=0;\ k=1,\ 2,\ 3 \tag{6}$$

in which $\beta\neq 0$ for $k=2$ and $\beta=0$ for $k=1, 3$.

By the geometric and material symmetry with respect to the cracked plane, it suffices to consider only the upper half region of the problem, $y\geq 0$, subjected to the homogeneous interface and boundary conditions imposed as

$$u_1(0, y)=u_2(h, y),\ v_1(0, y)=v_2(h, y)$$
$$u_2(0, y)=u_3(0, y),\ v_2(0, y)=v_3(0, y) \tag{7}$$

$$\sigma_{1xx}(0, y)=\sigma_{2xx}(h, y),$$
$$\sigma_{1xy}(0, y)=\sigma_{2xy}(h, y)$$
$$\sigma_{2xx}(0, y)=\sigma_{3xx}(0, y),\ \sigma_{2xy}(0, y)=\sigma_{3xy}(0, y) \tag{8}$$

$$u_1(x_1, y)\to 0,\ v_1(x_1, y)\to 0;\ x_1\to\infty \tag{9}$$

$$u_3(x_3, y)\to 0,\ v_3(x_3, y)\to 0;\ x_3\to -\infty \tag{10}$$

In addition, the symmetry and antisymmetry nature of the uncoupled mode I and mode II problems, respectively, stipulates that the separate sets of mixed conditions should be met along $y=0$ for each of the loading modes:

For mode I behavior;

$$\sigma_{1xy}(x_1, 0)=0;\ x_1\geq 0 \tag{11}$$

$$\sigma_{kxy}(x_k, 0)=0,\ v_k(x_k, 0)=0;\ k=2,\ 3,\quad 0\leq x_2\leq h,\ x_3\leq 0 \tag{12}$$

$$v_1(x_1, 0)=0;\ 0\leq x_1\leq a,\ x_1\geq b \tag{13}$$

$$\sigma_{1yy}(x_1, +0)=p(x_1);\ a\leq x_1\leq b \tag{14}$$

For mode II behavior;

$$\sigma_{1yy}(x_1, 0)=0;\ x_1\geq 0 \tag{15}$$

$$\sigma_{kyy}(x_k, 0)=0,\ u_k(x_k, 0)=0;\ k=2,\ 3,\quad 0\leq x_2\leq h,\ x_3\leq 0 \tag{16}$$

$$u_1(x_1, 0)=0;\ 0\leq x_1\leq a,\ x_1\geq b \tag{17}$$

$$\sigma_{1xy}(x_1, +0)=q(x_1);\ a\leq x_1\leq b \tag{18}$$

where $p(x_1)$ and $q(x_1)$ describe the arbitrary normal and shear crack surface tractions, respectively.

Based on the cosine/sine Fourier transform technique to solve Eqs. (5) and (6), the general expressions of displacement components for the cracked half-plane with $\beta=0$ and $(x, y)=(x_1, y)$, satisfying the regularity condition in Eq. (9), can be obtained as

$$u_1=\frac{2}{\pi}\int_0^\infty\left[A_1+A_2\left(x+\frac{\kappa}{s}\right)\right]e^{-sx}\begin{Bmatrix}\cos sy\\ -\sin sy\end{Bmatrix}ds+\frac{1}{2\pi}\int_{-\infty}^\infty(A_3+A_4y)\,e^{-|s|y-isx}ds \tag{19}$$

$$v_1=\frac{2}{\pi}\int_0^\infty(A_1+A_2x)\,e^{-sx}\begin{Bmatrix}\sin sy\\ \cos sy\end{Bmatrix}ds-\frac{i}{2\pi}\int_{-\infty}^\infty\frac{s}{|s|}\left[A_3+A_4\left(y+\frac{\kappa}{|s|}\right)\right]e^{-|s|y-isx}ds$$

$$;x \geq 0 \tag{20}$$

where s is the transform variable, $A_j(s)$, $j=1, .., 4$, are the arbitrary unknowns, and the upper/lower ones in the braces are for the mode I/II behavior, respectively.

For the graded interlayer with $\beta \neq 0$ and $(x, y)=(x_2, y)$, the general solutions of displacement components are also obtained as

$$u_2=\frac{2}{\pi}\int_0^{\infty}\sum_{j=1}^{4} B_j m_j e^{n_j x}\begin{Bmatrix}\cos sy\\ -\sin sy\end{Bmatrix}ds \tag{21}$$

$$v_2=\frac{2}{\pi}\int_0^{\infty}\sum_{j=1}^{4} B_j e^{n_j x}\begin{Bmatrix}\sin sy\\ \cos sy\end{Bmatrix}ds\ ;0\leq x\leq h \tag{22}$$

where $B_j(s)$, $j=1,..,4$, are the arbitrary unknowns, $n_j(s)$, $j=1,..,4$, are the roots of the characteristic equation:

$$(n^2+\beta n-s^2)^2+\left(\frac{3-\kappa}{1+\kappa}\right)\beta^2 s^2=0 \tag{23}$$

from which it can be shown that

$$n_j=-\frac{1}{2}\left[\beta+\sqrt{\beta^2+4s^2+(-1)^j 4\beta s i\left(\frac{3-\kappa}{1+\kappa}\right)^{1/2}}\right]$$

$$;\ \mathrm{Re}(n_j)<0,\ j=1,\ 2 \tag{24}$$

$$n_j=-\frac{1}{2}\left[\beta-\sqrt{\beta^2+4s^2-(-1)^j 4\beta s i\left(\frac{3-\kappa}{1+\kappa}\right)^{1/2}}\right]$$

$$;\ \mathrm{Re}(n_j)>0,\ j=3,\ 4 \tag{25}$$

and $m_j(s)$, $j=1,.., 4$, are given as

$$m_j=\frac{(\kappa-1)(n_j^2+\beta n_j)-(\kappa+1)s^2}{[2n_j+(\kappa-1)\beta]s} \tag{26}$$

For the half-plane without a crack and $\beta=0$ and $(x,y)=(x_3,y)$, the general solutions of displacement components that satisfy the regularity condition in Eq. (10) can be expressed as

$$u_3=-\frac{2}{\pi}\int_0^{\infty}\left[C_1+C_2\left(x-\frac{\kappa}{s}\right)\right]e^{sx}\begin{Bmatrix}\cos sy\\ -\sin sy\end{Bmatrix}ds \tag{27}$$

$$v_3=\frac{2}{\pi}\int_0^{\infty}(C_1+C_2 x)\,e^{sx}\begin{Bmatrix}\sin sy\\ \cos sy\end{Bmatrix}ds\ ;x\leq 0 \tag{28}$$

where $C_j(s)$, $j=1, 2$, are the arbitrary unknowns.

For each loading mode, there exist a total of ten arbitrary unknowns, A_j, B_j, $j=1,..,4$, and C_j, $j=1, 2$, in the general solutions of the elasticity equations. It is observed that the conditions in Eq. (12) and those in Eq. (16) for the mode I and mode II loadings, respectively, are satisfied by the general solutions as given in Eqs. (21), (22), (27), and (28). From the geometric and material features of the problem, it is noted that the mode I loading does not affect the mode II behavior, and vice versa.

3. Singular Integral Equations

In solving the current crack problem, the unknown auxiliary functions can be defined to replace the mixed conditions in Eqs. (13) and (14) for the mode I crack and those in Eqs. (17) and (18) for the mode II crack as

$$\phi_1(x_1)=\frac{\partial}{\partial x_1}v_1(x_1,\ 0)\ ;\ x_1\geq 0 \tag{29}$$

$$\phi_2(x_1)=\frac{\partial}{\partial x_1}u_1(x_1,\ 0)\ ;\ x_1\geq 0 \tag{30}$$

under the following conditions for the single-valuedness of displacements outside the crack region:

$$\phi_k(x_1)=0\ ;\ 0\leq x_1\leq a,\ x_1\geq b,\ k=1,\ 2 \tag{31}$$

$$\int_a^b \phi_k(x_1)\,dx_1=0\ ;\ k=1,\ 2 \tag{32}$$

As a result, for the mode I behavior, by using Eq. (11) and substituting from Eq. (20) into Eqs. (29) and (31), the unknowns A_j, $j=3, 4$, can be expressed in terms of ϕ_1 as

$$A_3(s)=\frac{1-\kappa}{2|s|}A_4(s)$$

$$=-\frac{1}{|s|}\frac{1-\kappa}{1+\kappa}\int_a^b \phi_1(t)\,e^{ist}dt \tag{33}$$

and for the mode II behavior, similarly from Eqs. (15), (19), (30), and (31), the expressions for A_j, $j=3, 4$, are written in terms of ϕ_2 such that

$$A_3(s)=-\frac{1+\kappa}{2|s|}A_4(s)$$

$$=-\frac{1}{is}\int_a^b \phi_2(t)\,e^{ist}dt \tag{34}$$

The remaining unknowns A_j, $j=1, 2$, B_j, $j=1,.., 4$, and C_j, $j=1, 2$, can also be determined in terms of ϕ_1 for mode I response and ϕ_2 for mode II response, by applying the interface conditions in Eqs. (7) and (8). The auxiliary functions ϕ_j, $j=1, 2$, thus become the only unknowns to be determined from the crack surface conditions in Eqs. (14) and (18), respectively, subjected to the compatibility condition in Eq. (32).

The general expressions for the traction components, σ_{1yy} and σ_{1xy}, are obtainable by substituting Eqs. (19) and (20) into Eqs. (3) and (4). Thereafter, upon using A_j, $j=3$, 4, as given above and A_j, $j=1$, 2 (see Appendix), followed by some algebraic manipulations, the traction components acting along the cracked plane, $y=0$, can be written as

$$\frac{\pi(1+\chi)}{2\mu_1}\lim_{y\to+0}\sigma_{1yy}(x,\ y)$$
$$=-i\int_a^b\phi_1(t)\,dt\int_{-\infty}^{\infty}\mathrm{sgn}(s)\,e^{is(t-x)}ds$$
$$+2\mu_1\int_a^b\phi_1(t)\,dt\int_0^{\infty}\Lambda_1(s,x,t)\,e^{-s(t+x)}ds$$
$$;0\leq x<\infty \qquad (35)$$
$$\frac{\pi(1+\chi)}{2\mu_1}\lim_{y\to+0}\sigma_{1xy}(x,\ y)$$
$$=-i\int_a^b\phi_2(t)\,dt\int_{-\infty}^{\infty}\mathrm{sgn}(s)\,e^{is(t-x)}ds$$
$$+2\mu_1\int_a^b\phi_2(t)\,dt\int_0^{\infty}\Lambda_2(s,x,t)\,e^{-s(t+x)}ds$$
$$;0\leq x<\infty \qquad (36)$$

where the improper integrals in the first terms on the right-hand side can be evaluated by employing the Fourier representation of a generalized function (Friedman, 1969)

$$\int_{-\infty}^{\infty}\mathrm{sgn}(s)\,e^{is(t-x)}ds=\frac{2i}{t-x} \qquad (37)$$

and the integrands $\Lambda_j(s,x,t)$, $j=1,2$, are expressed as

$$\Lambda_1(s,\ x,\ t)=\frac{1}{x}\left(t-\frac{1-\chi}{2s}\right)M_{11}(s,\ x)$$
$$+\frac{1}{x}\left(\frac{1+\chi}{2s}-t\right)M_{12}(s,\ x)$$
$$+2stM_{13}(s,\ x)+2(1-st)\,M_{14}(s,\ x) \qquad (38)$$
$$\Lambda_2(s,\ x,\ t)=\frac{1}{x}\left(t+\frac{1+\chi}{2s}\right)M_{21}(s,\ x)$$
$$+\frac{1}{x}\left(\frac{1-\chi}{2s}+t\right)M_{22}(s,\ x)$$
$$+2(1+st)\,M_{23}(s,\ x)+2stM_{24}(s,\ x) \qquad (39)$$

in which the functions $M_{lm}(s,\ x)$, $l=1$, 2, $m=1$,.., 4, are also given in the Appendix.

It is noted that for the crack tip away from the nominal interface with the graded interlayer, i. e., $a>0$, the integrands in Eqs. (38) and (39) possess the asymptotic behavior for large values of s such that

$$\lim_{s\to\infty}\Lambda_j(s,\ x,\ t)\,e^{-s(t+x)}=0$$
$$;a\leq(x,\ t)\leq b,\ j=1,\ 2 \qquad (40)$$

implying that the corresponding kernels in Eqs. (35) and (36) are bounded for all values of x and t in the closed domain $[a\ b]$. On the other hand, for the crack tip intersecting the interface with the interlayer as $a=0$, it can be shown that the kernels contain the logarithmic singularities as x and t approach zero simultaneously. Such unbounded terms are, however, square-integrable that do not affect the singular nature of the solution and can thus be treated as part of the regular kernels in the presence of the Cauchy singularity $1/(t-x)$ (Erdogan et al., 1991a). Accordingly, the near-tip stress field would be yet governed by the usual square-root singularity, provided the elastic properties are simply continuous and at least piecewise differentiable near and at the crack tip (Schovanec and Walton, 1988; Martin, 1992; Jin and Noda, 1994). These notable features are in contrast to the dependence of the order of the stress singularity on the elastic constants of the constituents when the crack tip terminates at the interface in piecewise homogeneous bonded media (Romeo and Ballarini, 1995).

As a consequence, with the imposition of the crack surface conditions in Eqs. (14) and (18), the singular integral equation for each of individual loading modes can be derived as

$$\int_a^b\frac{\phi_j(t)}{t-x}dt+\int_a^b k_j(x,\ t)\,\phi_j(t)\,dt$$
$$=f_j(x)\ ;\ a\leq x\leq b,\ j=1,\ 2 \qquad (41)$$

where $k_j(x,\ t)$, $j=1$, 2, are the regular Fredholm kernels and $f_j(x)$, $j=1$, 2, are the forcing terms written as

$$k_j(x,\ t)=\mu_1\int_0^{\infty}\Lambda_j(s,\ x,\ t)\,e^{-s(t+x)}ds\ ;\ j=1,\ 2 \qquad (42)$$

$$f_1(x)=\frac{\pi(1+\chi)}{4\mu_1}p(x),\ f_2(x)=\frac{\pi(1+\chi)}{4\mu_1}q(x) \qquad (43)$$

Because the two integral equations are not coupled, they are to be solved separately. In the crack problem under consideration, with the dominant Cauchy kernel in Eq. (41) solely contributing to the singular nature of the functions ϕ_j, $j=1, 2$, for $a=0$ as well as $a>0$, the square-root crack-tip behavior can be well retained in the form as (Muskhelishvili, 1953)

$$\phi_j(t)=\frac{g_j(t)}{\sqrt{(t-a)(b-t)}};\ a<t<b,\ j=1, 2 \qquad (44)$$

where $g_j(t)$, $j=1,2$, are the new unknown functions bounded and nonzero at $t=a$ and $t=b$, and in the normalized interval

$$\begin{Bmatrix} t \\ x \end{Bmatrix}=\frac{b-a}{2}\begin{Bmatrix} \eta \\ \xi \end{Bmatrix}+\frac{b+a}{2};\ -1<(\eta,\ \xi)<1 \qquad (45)$$

the solutions to the integral equations can therefore be expanded into the series of the orthogonal functions which in this case correspond to the Chebyshev polynomials of the first kind T_n as

$$\phi_j(t)=\phi_j(\eta)=\frac{1}{\sqrt{1-\eta^2}}\sum_{n=0}^{\infty} c_{jn}T_n(\eta)$$
$$;\ |\eta|<1,\ j=1, 2 \qquad (46)$$

in which c_{jn}, $j=1,2$, $n\geq 0$, are the unknown coefficients to be evaluated and the compatibility condition in Eq. (32) can be satisfied by using the orthogonality of T_n when $c_{j0}=0$, $j=1, 2$.

Upon substituting Eqs. (44)-(46) into Eq. (41), truncating the series at $n=N$, and using the relevant integral formula (Abramowitz and Stegun, 1972), the integral equations are rewritten as

$$\sum_{n=1}^{N} c_{jn}\left[\pi U_{n-1}(\xi)+\frac{b-a}{2}\int_{-1}^{1}\frac{k_j(\xi,\ \eta)\,T_n(\eta)}{\sqrt{1-\eta^2}}d\eta\right]$$
$$=f_j(\xi);\ j=1,\ 2,\ |\xi|<1 \qquad (47)$$

where U_n is the Chebyshev polynomial of the second kind. To solve the above functional equations, the zeros of $T_N(\xi)$ which are concentrated near the end singular points $\xi=\pm 1$ are chosen as a set of collocation points:

$$T_N(\xi_k)=0,\ \xi_k=\cos\left[\frac{\pi}{2}\frac{(2k-1)}{N}\right]$$
$$;\ k=1,\ 2,..,\ N \qquad (48)$$

and the integral equation for each loading mode can be then reduced to a system of linear algebraic equations for c_{jn}, $j=1, 2$, $1\leq n\leq N$, by evaluating Eq. (47) at N station points ξ_k. As noted, the number of series terms N which is equal to that of collocation points must be sufficiently large for the solution to converge to a required degree of accuracy.

With the coefficients c_{jn}, $j=1,2$, $1\leq n\leq N$, determined, the integral equations in Eq. (41) provide the dominant terms of singular tractions ahead of the crack tips $|\xi|>1$ as

$$\begin{Bmatrix} \sigma_{1yy}(\xi,\ 0) \\ \sigma_{1xy}(\xi,\ 0) \end{Bmatrix}=-\frac{4\mu_1}{1+\chi}\sum_{n=1}^{N}\begin{Bmatrix} c_{1n} \\ c_{2n} \end{Bmatrix}$$
$$\frac{[\xi-\mathrm{sgn}(\xi)\sqrt{\xi^2-1}]^n}{\mathrm{sgn}(\xi)\sqrt{\xi^2-1}}+O(1);\ |\xi|>1 \qquad (49)$$

where $O(\cdot)$ denotes the nonsingular terms involving the Fredholm kernels. Subsequently, as the physical quantities of primary importance in characterizing the local crack-tip behavior in elastic fracture mechanics, the stress intensity factors are defined that can be evaluated in terms of the solutions to the integral equations such that

$$\begin{Bmatrix} K_{\mathrm{I}}(a) \\ K_{\mathrm{II}}(a) \end{Bmatrix}\equiv\lim_{x\to a}\sqrt{2(a-x)}\begin{Bmatrix} \sigma_{1yy}(x,\ 0) \\ \sigma_{1xy}(x,\ 0) \end{Bmatrix};\ x<a$$
$$=\frac{4\mu_1}{1+\chi}\sqrt{\frac{b-a}{2}}\sum_{n=1}^{N}(-1)^n\begin{Bmatrix} c_{1n} \\ c_{2n} \end{Bmatrix} \qquad (50)$$

$$\begin{Bmatrix} K_{\mathrm{I}}(b) \\ K_{\mathrm{II}}(b) \end{Bmatrix}\equiv\lim_{x\to b}\sqrt{2(x-b)}\begin{Bmatrix} \sigma_{1yy}(x,\ 0) \\ \sigma_{1xy}(x,\ 0) \end{Bmatrix};\ x>b$$
$$=-\frac{4\mu_1}{1+\chi}\sqrt{\frac{b-a}{2}}\sum_{n=1}^{N}\begin{Bmatrix} c_{1n} \\ c_{2n} \end{Bmatrix} \qquad (51)$$

where K_{I} and K_{II} are the uncoupled mode I and mode II stress intensity factors, respectively. To be mentioned now is that due to the continuity of the elastic properties through the graded interlayer, the singular stress components as given in Eqs. (49) are also continuous at the location of the nominal interface with the interlayer, which renders the foregoing definition of the stress intensity factors equally applicable even for the limiting case of $a=0$ as well.

4. Results and Discussion

As numerical illustrations, the stress intensity factors are presented for various combinations of geometric (d/c, $h/2c$) and material (E_3/E_1) parameters of the bonded half-planes with a graded interlayer, in order to cover a wide range of possibilities that may arise in practice depending upon particular applications. Without loss in generality, the uniform normal and shear crack surface tractions are considered to be applied as $p(x) = -\sigma_0$, $q(x) = 0$ and $p(x) = 0$, $q(x) = -\tau_0$ in Eqs. (14) and (18). The plane strain condition is assumed with the Poisson's ratio $\nu = 0.3$, unless otherwise stated and the resulting values of the mode I and mode II stress intensity factors are presented in normalized form as $K_I/\sigma_0 c^{1/2}$ and $K_{II}/\tau_0 c^{1/2}$, respectively. In the course of generating the numerical results, no more than thirty-term expansion in Eq. (46) is found to be necessary in obtaining three-digit accuracy beyond the decimal point for the material and geometric configurations examined in this study, together with the related integrals in Eqs. (42) and (47) evaluated by using the Gauss-Legendre and Gauss-Chebyshev quadratures, respectively (Davis and Rabinowitz, 1984).

Figures 2 and 3 show the variations of mode I and mode II stress intensity factors, respectively, as a function of crack location d/c for different values of the elastic moduli ratio E_3/E_1 and the fixed interlayer thickness, $h/2c=0.5$. A common feature to be observed from these two figures is that both the values of K_I and K_{II} decrease with increasing E_3/E_1 which can be accounted for by the overall stiffness augmentation in the bonded system, indicating the enhanced fracture resistance for the crack in a compliant constituent by the nearby stiff constituent. An additional generic trend is that both the stress intensity factors also tend to decrease with decreasing d/c for $E_3/E_1 > 1.0$, implying the more constraints exerted by the stiff constituent when the crack approaches the interlayer, while the reverse behavior prevails for $E_3/E_1 < 1.0$.

It is further noted that for the crack in a stiffer constituent, the stress intensity factors at the crack tip a are of greater magnitude than those at the crack tip b, and the opposite is true for the case of a crack in the less stiff constituent. As expected, the crack tip closer to the interface is shown to be more markedly affected by the values of E_3/E_1. When the crack is being located farther away from the interface, however, the normalized stress intensity factors at both the crack tips approach unity, i. e., those of a crack in an infinite homogeneous plane. Under the identical geometric condition, a careful examination of the curves for K_I and K_{II} reveals that although not so pronounced, K_{II} is less sensitive to the variations of E_3/E_1 than K_I is.

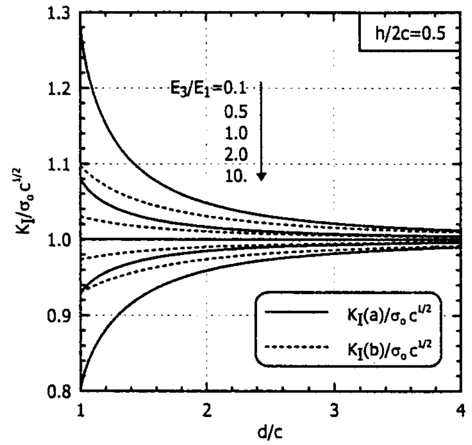

Fig. 2 Variations of mode I stress intensity factors $K_I/\sigma_0 c^{1/2}$ with crack location d/c for different elastic moduli ratios E_3/E_1 ($h/2c=0.5$)

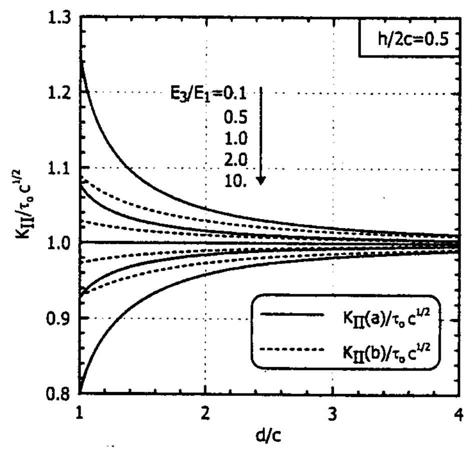

Fig. 3 Variations of mode II stress intensity factors $K_{II}/\tau_0 c^{1/2}$ with crack location d/c for different elastic moduli ratios E_3/E_1 ($h/2c=0.5$).

To examine the effect of the graded interlayer, the mode I and mode II stress intensity factors for different values of $h/2c$ are plotted versus d/c in Figs. 4 and 5, respectively. Two material combinations, $E_3/E_1=0.1$ and $E_3/E_1=10$, are considered. Of importance in these figures is that an increase in $h/2c$ leads to a substantial reduc-

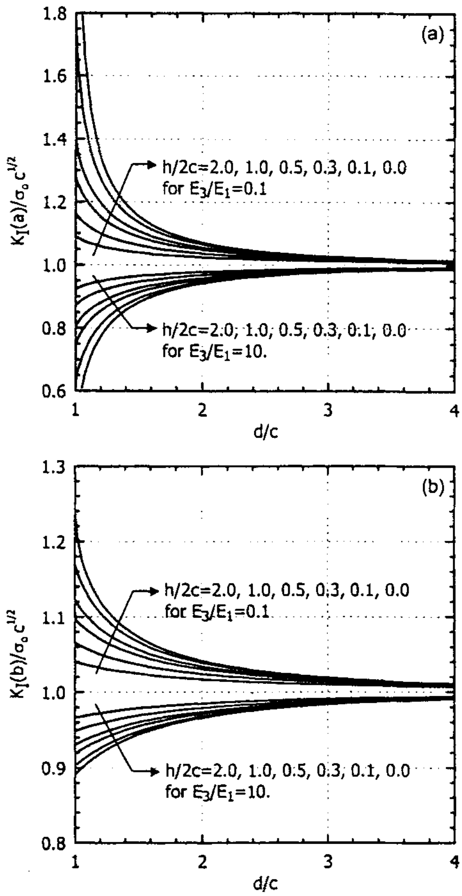

Fig. 4 Variations of mode I stress intensity factors $K_{\mathrm{I}}/\sigma_0 c^{1/2}$ (a) at the crack-tip a; (b) at the crack-tip b with crack location d/c for different interlayer thicknesses $h/2c$ ($E_3/E_1=0.1$ and $E_3/E_1=10$).

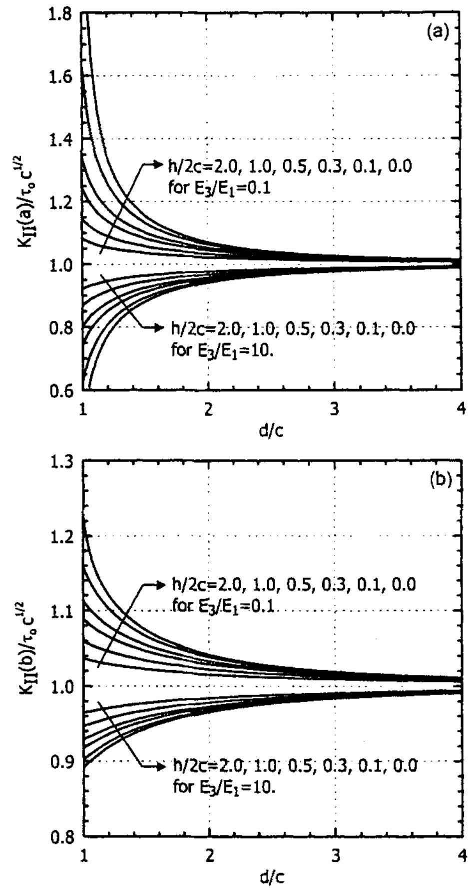

Fig. 5 Variations of mode II stress intensity factors $K_{\mathrm{II}}/\tau_0 c^{1/2}$ (a) at the crack-tip a; (b) at the crack-tip b with crack location d/c for different interlayer thicknesses $h/2c$ ($E_3/E_1=0.1$ and $E_3/E_1=10$).

tion in the magnitude of both modes of stress intensity factors when the crack is in the stiffer constituent and the opposite response is seen to exist for the reversed material combination, which is quite similar to increasing the value of d/c. Such a trend with $h/2c$, including the limiting case of $h/2c=0.0$, signifies that for the crack in the stiff constituent, the presence of a graded interlayer with the greater thickness provides more constraints which are effective in shielding the crack, offsetting the influence from the adjacent compliant constituent.

To be noticed herein is that for $h/2c=0.0$, the case of a bimaterial system without the interlayer, the corresponding results for the crack tip a are plotted for $d/c \geq 1.05$. This is due to the fact that the discrete nature of the elastic moduli for zero interlayer thickness yields the order of singularity at the crack tip a other than the square-root type when the crack intersects the bimaterial interface (Romeo and Ballarini, 1995). Hence, the values of stress intensity factors at the crack tip a for the current trimaterial and conventional bimaterial systems are not compatible when $d/c=1.0$. Otherwise, a continuous transition of both the mode I and mode II stress intensity factors is retained for the given values of $h/2c$. If the thickness ratio $h/2c$ were increased even further, the dilution of degree of nonhomogeneity within the interlayer would lead to the solutions that also tend to those for the homogeneous counterpart. Another feature to be closely observed from Figs. 4 and 5 is that the behavior of the mode I crack appears to be more sensitive to changes in the interlayer thickness when compared with that of the mode II crack, especially for the crack in the stiffer constituent.

Additional results are presented in Figs. 6 and 7 in order to give some ideas about the effect of variations in the Poisson's ratio ν on the modes I and II stress intensity factors, respectively. Specifically, the values of the stress intensity factors are plotted in the logarithmic scale with the variable E_3/E_1 for different values of ν and the fixed crack size and location as $h/2c=0.5$ and $d/c=1.0$. Although seemingly insignificant, the effect of ν is observed to become more notable when the crack is in the stiffer constituent. The difference in the magnitude of stress intensity factors for the given range of the Poisson's ratio, $0.01\leq\nu\leq0.49$, is estimated to be as high as 2.1 percent for the mode I crack and 4.3 percent for the mode II crack when $E_3/E_1=0.1$. It is practically likely, however, that the values of ν vary within a much narrower range than the above so that the effect of the Poisson's ratio may be negligible. Therefore, analogous to the previous findings for the case of a single crack (Konda and Erdogan, 1994) and that of a periodic array of parallel cracks (Choi, 1997) in the graded nonhomogeneous plane, the assumption of neglecting the possible spatial variation of the Poisson's ratio in the analysis of bonded media with the graded interfacial zone is too justifiable.

Of interest in Figs. 6 and 7 is that both the mode I and mode II stress intensity factors vary approximately linearly in the logarithmic scale with the elastic moduli ratio E_3/E_1 under the given crack surface tractions. Because the effect of the Poisson's ratio is negligibly small, it is then possible to develop simple empirical formulas for the stress intensity factors. For $\nu=0.3$ from Figs. 6 and 7, based on a power relationship, the following approximate formulas can be used to evaluate both modes of the stress intensity factors for $h/2c=0.5$ and $d/c=1.0$:

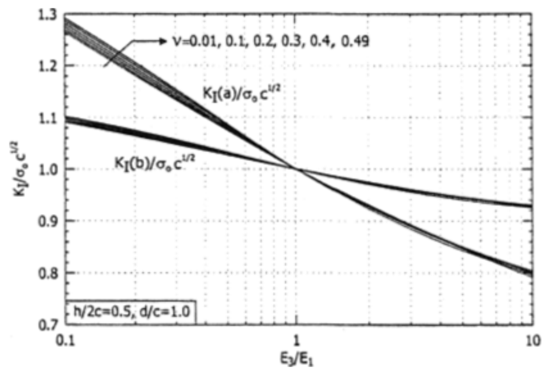

Fig. 6 Effect of the Poisson's ratio ν on the values of mode I stress intensity factors $K_I/\sigma_0 c^{1/2}$ versus elastic moduli ratios E_3/E_1 ($h/2c=0.5$ and $d/c=1.0$)

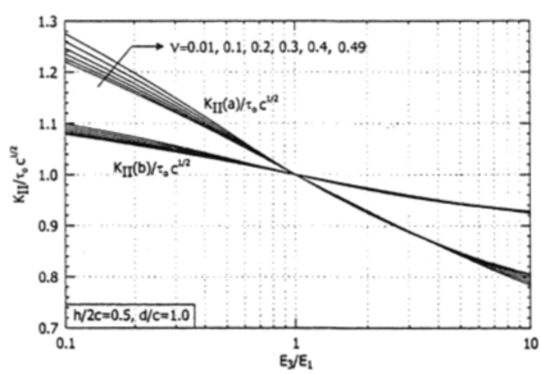

Fig. 7 Effect of the Poisson's ratio ν on the values of mode II stress intensity factors $K_{II}/\tau_0 c^{1/2}$ versus elastic moduli ratios E_3/E_1 ($h/2c=0.5$ and $d/c=1.0$)

$$\frac{K_I(a)}{\sigma_0\sqrt{c}}\cong\begin{cases}(E_3/E_1)^{-0.10755}; & 0.1\leq E_3/E_1\leq1.0\\(E_3/E_1)^{-0.10624}; & 1.0\leq E_3/E_1\leq10.\end{cases}\tag{52}$$

$$\frac{K_I(b)}{\sigma_0\sqrt{c}}\cong\begin{cases}(E_3/E_1)^{-0.04044}; & 0.1\leq E_3/E_1\leq1.0\\(E_3/E_1)^{-0.03574}; & 1.0\leq E_3/E_1\leq10.\end{cases}\tag{53}$$

$$\frac{K_{II}(a)}{\tau_0\sqrt{c}}\cong\begin{cases}(E_3/E_1)^{-0.09573}; & 0.1\leq E_3/E_1\leq1.0\\(E_3/E_1)^{-0.10791}; & 1.0\leq E_3/E_1\leq10.\end{cases}\tag{54}$$

$$\frac{K_{II}(b)}{\tau_0\sqrt{c}}\cong\begin{cases}(E_3/E_1)^{-0.03707}; & 0.1\leq E_3/E_1\leq1.0\\(E_3/E_1)^{-0.03386}; & 1.0\leq E_3/E_1\leq10.\end{cases}\tag{55}$$

where another similar empirical relationships for other values of $h/2c$ and d/c can be also readily developed.

5. Closing Remarks

The plane elasticity solutions have been provided for the problem of a crack perpendicular to the graded interfacial zone in bonded materials under both the mode I and mode II loading conditions. With the interfacial zone treated as a nonhomogeneous interlayer between the dissimilar homogeneous half-planes, a singular integral equation was derived for each mode of the crack-tip behavior. The main emphasis was placed upon the evaluation of stress intensity factors from the standard disciplines of linear elastic fracture mechanics. The mode I and mode II crack-tip behaviors were then described in terms of the corresponding values of stress intensity factors that were shown to be strongly affected by the relative crack size and location for different combinations of elastic moduli of bonded media involving the material nonhomogeneity within the interlayer. The effect of such a graded interlayer was also addressed in conjunction with the results for the case of bimaterial interface of zero thickness in piecewise homogeneous bonded materials.

It was further illustrated that the presence of the graded interlayer plays the role of shielding or amplifying the near-tip field, depending on the material combination of the bonded trimaterial system. To be specific, for the crack in the stiffer side of the bonded media, the graded interlayer with the greater thickness would be preferable attenuating the amplifying effect. On the other hand, for the case of a crack residing in the less stiff side, the interlayer with the smaller thickness would be more effective in curbing the crack-tip behavior.

It should be also noted that although the mode I and mode II crack responded to the variations of material and geometric parameters in a comparable way, the values of the mode I stress intensity factor were observed to be more sensitive to the changes in such parameters than those of the mode II stress intensity factor. Moreover, the insignificant effect of the Poisson's ratio led to some approximate formulas that can readily be used to evaluate the stress intensity factors as a function of the elastic moduli ratio of the bonded dissimilar half-planes with the graded interlayer.

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Appendix

Under each loading mode, the remaining eight unknowns A_j, $j=1, 2$, B_j, $j=1,.., 4$, and C_j, $j=1, 2$, in Eqs. (19)-(22) and Eqs. (27) and (28) can be determined by applying the interface conditions in Eqs. (7) and (8). In particular, the unknowns A_j, $j=1, 2$, required in deriving the integral equations are obtained for the mode I and mode II behaviors in the form as

$$\begin{Bmatrix} A_1(s) \\ A_2(s) \end{Bmatrix}_{\text{mode I}} = \frac{\mu_1}{\chi(1+\chi)} \sum_{j=1}^{2} \begin{Bmatrix} H_{1j} \\ H_{1(j+2)} \end{Bmatrix} g_{1j}(s) + \frac{2\mu_1}{1+\chi} \sum_{j=3}^{4} \begin{Bmatrix} H_{1(j+2)} \\ H_{1(j+4)} \end{Bmatrix} g_{1j}(s) \tag{A1}$$

$$\begin{Bmatrix} A_1(s) \\ A_2(s) \end{Bmatrix}_{\text{mode II}} = \frac{\mu_1}{\chi(1+\chi)} \sum_{j=1}^{2} \begin{Bmatrix} H_{2j} \\ H_{2(j+2)} \end{Bmatrix} g_{2j}(s) + \frac{2\mu_1}{1+\chi} \sum_{j=3}^{4} \begin{Bmatrix} H_{2(j+2)} \\ H_{2(j+4)} \end{Bmatrix} g_{2j}(s) \tag{A2}$$

where $H_{lm}(s)$, $l=1, 2$, $m=1,.., 8$, are certain intricate functions of the elastic moduli, the geometry of the bonded trimaterial system, and the Fourier transform variable s as well, together with the functions $g_{lm}(s)$, $l=1, 2$, $m=1,.., 4$, expressed in terms of the auxiliary functions ϕ_j, $j=1, 2$, as

$$g_{11}(s) = \int_a^b \left(t - \frac{1-\chi}{2s} \right) e^{-st} \phi_1(t)\, dt \tag{A3}$$

$$g_{12}(s) = \int_a^b \left(\frac{1+\chi}{2s} - t \right) e^{-st} \phi_1(t)\, dt \tag{A4}$$

$$g_{13}(s) = \int_a^b ste^{-st} \phi_1(t)\, dt \tag{A5}$$

$$g_{14}(s) = \int_a^b (1-st)\, e^{-st} \phi_1(t)\, dt \tag{A6}$$

$$g_{21}(s) = \int_a^b \left(t + \frac{1+\chi}{2s} \right) e^{-st} \phi_2(t)\, dt \tag{A7}$$

$$g_{22}(s) = \int_a^b \left(\frac{1-\chi}{2s} + t \right) e^{-st} \phi_2(t)\, dt \tag{A8}$$

$$g_{23}(s)=\int_a^b (1+st)\,e^{-st}\phi_2(t)\,dt \qquad \text{(A9)}$$

$$g_{24}(s)=\int_a^b ste^{-st}\phi_2(t)\,dt \qquad \text{(A10)}$$

When Eqs. (A1) and (A2) are substituted into the expressions for the traction components σ_{1yy} and σ_{1xy} defined along the cracked plane, $y=0$, the functions $M_{lm}(s, x)$, $l=1, 2$, $m=1,.., 4$, in Eqs. (38) and (39) can be written as

$$M_{1m}(s, x)=sH_{1m}+\left(sx-\frac{3}{2}+\frac{x}{2}\right)H_{1(m+2)}$$
$$;\ m=1, 2 \qquad \text{(A11)}$$

$$M_{1m}(s, x)=sH_{1(m+2)}+\left(sx-\frac{3}{2}+\frac{x}{2}\right)H_{1(m+4)}$$
$$;\ m=3, 4 \qquad \text{(A12)}$$

$$M_{2m}(s, x)=-sH_{2m}-\left(sx-\frac{1}{2}+\frac{x}{2}\right)H_{2(m+2)}$$
$$;\ m=1, 2 \qquad \text{(A13)}$$

$$M_{2m}(s, x)=-sH_{2(m+2)}-\left(sx-\frac{1}{2}+\frac{x}{2}\right)H_{2(m+4)}$$
$$;\ m=3, 4 \qquad \text{(A14)}$$